Quasi-One-Dimensional Method for Calculating Asymmetric Flows Through Nozzles

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The asymmetric steady-state inviscid adiabatic flow of a perfect gas through a subsonic/supersonic nozzle is considered. The nozzle itself is either of rectangular cross section or is axisymmetric. A first-order small asymmetry induced by an entrance flow that is oblique to the nozzle axis and has a transverse pressure gradient is allowed. The cross section of the nozzle is assumed to vary only slowly as a function of the axial distance. The case considered here is, therefore, an extension of the well-known theory of the quasi-one-dimensional flow of a perfect gas treated in standard textbooks. An integral method is used to obtain approximate results. The method is simple, yet in a test case (supersonic flow at an angle of attack through a rectangular channel) where the exact first-order result is known, a comparison shows surprisingly good agreement.

Nomenclature

a, b = parameters [Eq. (14)]
$F_1(z)$ = transverse force on nozzle, per unit axial length
g_1, k_1 = functions characterizing transverse pressure gradient [Eqs. (7) and (8)]
h = enthalpy
$L_{1y}(z)$ = transverse component of angular momentum
$M_0(z)$ = Mach number of zero-order (unperturbed) flow
$P_{1x}(z)$ = transverse component of momentum
p = pressure
$R(z)$ = half-width of rectangular nozzle and radius of axisymmetric nozzle
u, v, w = Cartesian velocity components
x, y, z; r, ϕ, z = Cartesian and cylindrical coordinates
γ = ratio of specific heats
ξ, ζ = nondimensional lengths [Eq. (8)]
ρ = density
σ = $\sigma = 1$ for rectangular nozzle; $\sigma = 2$ for axisymmetric nozzle
$(\ )_0$ = zero-order term
$(\ )_1$ = first-order asymmetric term
$(\ )^*$ = condition at $M_0 = 1$
$(\ )_i$ = initial condition
$(\ )_e$ = nozzle exit plane condition

I. Introduction

HIGH-SPEED flows that have a small lateral asymmetry occur in rocket motors that have canted nozzles (to avoid impingement of the plume on adjacent structures) or nozzles that can be vectored. Interest in this type of flow has been renewed recently, since it is thought to play a role in an instability observed in certain spin-stabilized solid-propellant upper-stage rockets. In this case, the flow at the nozzle entrance is oblique to the nozzle axis as a consequence of the Coriolis force acting on the combustion gas in a precessing and nutating rocket.[1,2] The case of spinning rockets is further discussed below.

Nonaxisymmetric flows through nozzles have also been considered by Walters,[3] both experimentally and theoretically. However, in that case, the nozzles themselves were nonaxisymmetric, having an obliquely machined throat section. Other authors have studied nonaxisymmetric flows in nozzles, but generally they have confined themselves to the supersonic part of the flow, computed by the method of characteristics, starting from an assumed inclined sonic surface at the nozzle throat.[4,5] None of these studies is directly applicable to the case encountered in canted nozzles or in spinning and precessing rocket vehicles.

A more extensive treatment of asymmetric flows in jet engines and rocket nozzles is found in Chapter 7 of the monograph by U. G. Pirumov and G. S. Roslyakov, which has recently become available in English translation.[6] Most important in their work is the application to nozzle sections that consist of truncated staight cones, or to several such sections joined together. A perturbation method is used, where the zero-order solution is the one pertaining to a purely radial flow with its source at the cone's apex. Profiled nozzles, which have curved contours, are studied numerically, and experiments are reported but are almost exclusively confined to the supersonic divergent section.

In these applications, the flow—as a correction to the axisymmetric flow—can be sufficiently approximated closely by assuming a steady-state, inviscid and adiabatic flow of a perfect gas. Based on the assumption of constant reservoir conditions upstream of the nozzle, the flow is isentropic and isoenergetic (constant total enthalpy) everywhere. Analogously to the well-known elementary theory of quasi-one-dimensional flow, the nozzle cross section is assumed to vary only slowly with axial distance. Since the lateral asymmetry caused by the angle of attack, or transverse pressure gradient, of the entrance flow is also assumed small, the effects due to the changing cross section and those due to the lateral asymmetry can both be treated as small perturbations superposed on the one-dimensional zero-order flow.

Rocket motors that are used in a spinning mode for upper-stage or spacecraft propulsion are usually of spheroidal shape and have a large ratio of effective motor diameter to throat diameter. For this reason, at the propellant burn surface, the kinetic energy of the axial and radial velocity components is very small compared with the enthalpy. For the rates of spin encountered in practice, the circumferential velocity—which, at the burn surface, conforms to a solid body rotation—is, at most, of the same order as the axial velocity; hence, its associate kinetic energy is also negligible. For these reasons, the usual assumption (which is used below) that the stagnation

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enthalpy is uniform is justified even for spin-stabilized rockets.

The approach used in this paper is based on a simple integral method and is, therefore, quite elementary. Methods of this type are better known in boundary-layer theory. They are difficult to justify on purely theoretical grounds, since, typically, the terms retained are not simply the lowest order terms of a convergent power series, particularly not in supersonic flows, which are fundamentally nonanalytic. Nevertheless, experience has shown that these methods, if applied with care, often give satisfactory results. In Sec. IV, we give an example of a flow where a first-order exact solution is known, and compare it with the result obtained by the integral method developed in this paper.

The analysis also applies to flows that are either entirely subsonic or entirely supersonic; but emphasis is placed on flows that have both a subsonic and supersonic part, without shocks, such as is the case for a normally operating Laval nozzle.

We choose Cartesian coordinates x,y,z for nozzles with rectangular cross sections and also polar coordinates r, ϕ, z for axisymmetric nozzles. In each case, the z-axis is the longitudinal axis, and the coordinate origin is taken at the point on the axis where the Mach number is unity. The half-width in the x-direction of the rectangular nozzle and the radius of the axisymmetric nozzle are both designated by $R(z)$. It will be found convenient to associate the numbers $\sigma = 1$ with the rectangular case and $\sigma = 2$ with the axisymmetric case.

The obliquity of the inlet flow gives rise to the asymmetric pressure term $\epsilon p_1(x,y,z)$, where $|\epsilon| \ll 1$. In turn, the slope of the nozzle wall generates a symmetric pressure term. Both are treated as perturbations superposed on the pressure $p_0(z)$ of the quasi-one-dimensional flow. We write, to first order

$$p = p_0(z) + \epsilon p_1 + \text{s.f.p.} \tag{1a}$$

and similarly for the density ρ, enthalpy h, and the Cartesian velocity components u, v, w; hence, for instance, for u we write

$$u = u_0(z) + \epsilon u_1 + \text{s.f.p.} \tag{1b}$$

The symmetric flow perturbation (s.f.p.) terms are written informally merely as a reminder; it will be clear from the subsequent development that, by reason of symmetry, they do not contribute to any of the integrals defined below.

The first-order asymmetric terms ρ_1, h_1, w_1 can all be expressed in terms of p_1. Thus, from the assumption of a perfect gas and constant entropy, $\rho/\rho_0(z) = [p/p_0(z)]^{1/\gamma}$, where γ is the ratio of the specific heats. Since this relation applies separately to the symmetric and the asymmetric case and superposition holds, it follows, after neglecting second- and higher-order terms, that

$$\rho_1/\rho_0(z) = \frac{1}{\gamma}\, p_1/p_0(z) \tag{2a}$$

and similarly

$$h_1/h_0(z) = \frac{\gamma - 1}{\gamma}\, p_1/p_0(z) \tag{2b}$$

From conservation of energy, $h + 1/2(u^2 + v^2 + w^2) = \text{const}$, and neglecting again second-order terms, it follows that $w_1/w_0 = -h_1/w_0^2$. Introducing the Mach number $M_0(z)$ of the quasi-one-dimensional flow, $M_0 = w_0/\sqrt{(\gamma - 1)h_0}$, we can express the asymmetric perturbation of the axial velocity component by

$$w_1/w_0(z) = -[\gamma M_0^2(z)]^{-1}\, p_1/p_0(z) \tag{2c}$$

II. Conservation Equations

We consider a control volume bounded by transverse planes at z and $z + dz$ and by the nozzle walls. We designate by $\epsilon P_{1x}(z)$ the transverse momentum (which is in the x-direction) carried by the flow through the transverse plane, per unit time. In the case of the rectangular nozzle, P_{1x} is taken per unit width in the y-direction. From conservation of momentum to the lowest significant order

$$\frac{dP_{1x}}{dz} = -[(p_1)_{x=R(z)} - (p_1)_{x=-R(z)}] \qquad \text{if } \sigma = 1 \tag{3a}$$

$$\frac{dP_{1x}}{dz} = -R(z)\int_{\phi=0}^{2\pi} (p_1)_{r=R(z)} \cos\phi \, d\phi \qquad \text{if } \sigma = 2 \tag{3b}$$

for the rectangular ($\sigma = 1$) and axisymmetric ($\sigma = 2$) nozzle, respectively.

The transverse angular momentum about the origin of the coordinate system is in the y-direction. The amount carried through a plane normal to the nozzle axis, per unit time (and per unit width in the case of the rectangular nozzle), is designated by $\epsilon L_{1y}(z)$. When second- and higher-order terms are dropped, the momentum flux ρw^2 becomes

$$\rho w^2 = \rho_0 w_0^2 + 2\epsilon\rho_0 w_0 w_1 + \epsilon w_0^2 \rho_1 + \text{s.f.p.}$$

By symmetry, the zero and symmetric first-order terms in ρw^2 make no contribution to the integration over x, so that for the rectangular nozzle

$$L_{1y} = zP_{1x} - \int_{x=-R(z)}^{R(z)} (2\rho_0 w_0 w_1 + w_0^2 \rho_1)\, x \, dx \qquad \text{if } \sigma = 1 \tag{4a}$$

The first term on the right comes from the u component of the velocity, integrated over the cross section; the second term is from w.

By essentially the same argument, for the axisymmetric nozzle

$$L_{1y} = zP_{1x} - \int_{r=0}^{R(z)} \int_{\phi=0}^{2\pi} (2\rho_0 w_0 w_1 + w_0^2 \rho_1)\, r^2 \cos\phi \, d\phi \, dr \qquad \text{if } \sigma = 2 \tag{4b}$$

Conservation of angular momentum then requires that

$$\frac{dL_{1y}}{dz} = \frac{d}{dz}\int_{x=-R(z)}^{R(z)} p_1 x \, dx + \left(z + R\frac{dR}{dz}\right)\frac{dP_{1x}}{dz} \qquad \text{if } \sigma = 1 \tag{5a}$$

$$\frac{dL_{1y}}{dz} = \frac{d}{dz}\int_{r=0}^{R(z)}\int_{\phi=0}^{2\pi} p_1 r^2 \cos\phi \, d\phi \, dr + \left(z + R\frac{dR}{dz}\right)\frac{dP_{1x}}{dz} \qquad \text{if } \sigma = 2 \tag{5b}$$

where Eqs. (3a) and (3b), respectively, have been used and where again the zero-order and first-order symmetric terms make no contribution to the integrals. The first term on the right represents the moment from forces acting on the transverse planes bounding the control volume; the last term results from the wall pressure.

The term L_{1y} is eliminated from these equations by differentiating Eqs. (4) and substituting the result into Eqs. (5). When

Eqs. (2) also are used to eliminate ρ_1 and w_1, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_{x=-R(z)}^{R(z)}(M_0^2-1)p_1 x\,\mathrm{d}x + R\,\frac{dR}{\mathrm{d}z}\frac{\mathrm{d}P_{1x}}{\mathrm{d}z} - P_{1x} = 0$$

if $\sigma = 1$

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_{r=0}^{R(z)}\int_{\phi=0}^{2\pi}(M_0^2-1)p_1 r^2\cos\phi\,\mathrm{d}\phi\,\mathrm{d}r$$

$$+ R\,\frac{dR}{\mathrm{d}z}\frac{\mathrm{d}P_{1x}}{\mathrm{d}z} - P_{1x} = 0 \qquad \text{if } \sigma = 2$$

where use has been made of the expression for the dynamic pressure of the quasi-one-dimensional flow[7] $(\rho_0/2)w_0^2 = (\gamma/2)p_0M_0^2$. Differentiating again with respect to z and using Eqs. (3) gives the following equations for p_1, where now all other first-order perturbation terms have been eliminated:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left\{(M_0^2-1)\int_{x=-R(z)}^{R(z)}p_1 x\,\mathrm{d}x\right\} - R\,\frac{\mathrm{d}R}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}z}$$

$$[(p_1)_{x=R(z)} - (p_1)_{x=-R(z)}] + \left[1 - \frac{\mathrm{d}}{\mathrm{d}z}\left(R\,\frac{\mathrm{d}R}{\mathrm{d}z}\right)\right]$$

$$[(p_1)_{x=R(z)} - (p_1)_{x=-R(z)}] = 0, \qquad \text{if } \sigma = 1 \tag{6a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left\{(M_0^2-1)\int_{r=0}^{R(z)}\int_{\phi=0}^{2\pi}p_1 r^2\cos\phi\,\mathrm{d}\phi\,\mathrm{d}r\right\} - R\,\frac{\mathrm{d}R}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}z}$$

$$\left[R\int_{\phi=0}^{2\pi}(p_1)_{r=R(z)}\cos\phi\,\mathrm{d}\phi\right] + \left[1 - \frac{\mathrm{d}}{\mathrm{d}z}\left(R\,\frac{\mathrm{d}R}{\mathrm{d}z}\right)\right]$$

$$R\int_{\phi=0}^{2\pi}(p_1)_{r=R(z)}\cos\phi\,\mathrm{d}\phi = 0, \qquad \text{if } \sigma = 2 \tag{6b}$$

III. Integral Method

Integral methods are typically based on prescribing for the dependent variable a simple functional form that may depend on one or several parameters and satisfies the boundary conditions. The parameters then are determined such that the integral relations—in this case, the conservation equations for the transverse momentum and angular momentum integrated over the nozzle cross section—are satisfied. With a judicious choice of the functional dependence, useful results, although of limited accuracy, can often be obtained.

In the present case, we prescribe for the asymmetric perturbation term p_1 a linear dependence on the transverse coordinate, in the form, therefore,

$$p_1/p_0(z) = g_1(z)x/R(z) \tag{7}$$

where the nondimensional coefficient $g_1(z)$ is to be determined.

We designate by R^* the nozzle half width ($\sigma = 1$) or nozzle radius ($\sigma = 2$) at the throat (a fictitious throat if the quasi-one-dimensional flow is subsonic throughout) where $M_0 = 1$, and $p_0 = p_0^*$ and define the nondimensional quantities

$$\xi = R/R^*, \qquad \zeta = z/R^*, \qquad k_1(\zeta) = \frac{R^*}{R(z)}\frac{p_0(z)}{p_0^*}\,g_1(z) \tag{8}$$

Carrying out the integrations in Eqs. (6) yields a combined relation for the rectangular and axisymmetric nozzles, in the form of a second-order differential equation for $k_1(\zeta)$

$$\frac{1}{\sigma+2}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\left[\xi^{\sigma+2}\left(M_0^2-1\right)k_1\right] - \frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\mathrm{d}\xi}{\mathrm{d}\zeta}\,\xi^{\sigma+1}\,k_1\right]$$

$$+\ \xi^{\sigma}\,k_1 = 0 \tag{9}$$

Equation (9) represents the principal result of Sections I through III, and is suitable for computer programming of the direct problem (prescribed zero order Mach number as a function of axial distance) or of the inverse problem (prescribed nozzle contour). In either case, it is advantageous to express the function $\xi(\zeta)$ in Eq. (9) in terms of the Mach number function $M_0(\zeta)$. From a well-known formula for quasi-one-dimensional flows[7]

$$\xi = \left\{\frac{1}{M_0^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_0^2\right)\right]^{(\gamma+1)/(\gamma-1)}\right\}^{1/(2\sigma)} \tag{10}$$

As one would expect, Eq. (9) exhibits the transition at $M_0 = 1$ between an essentially exponential behavior of $k_1(\zeta)$ and, on the other hand, a wavelike character as the sign of the coefficient of the highest (second) derivative changes.

As is well-known, classical one-dimensional nozzle theory requires the Mach number to be unity at a throat where a transition from subsonic to supersonic flow occurs. An analogous condition occurs in the present case and is caused by the vanishing of the coefficient in Eq. (9) of the second derivative of $k_1(\zeta)$ at $M_0 = 1$. Carrying out the differentiations indicated in Eq. (9) and letting at the throat $M_0 = 1$, $\xi = 1$, $\mathrm{d}\xi/\mathrm{d}\zeta = 0$ results in the condition for the solution to be regular at $\zeta = 0$

$$\frac{2}{\sigma+2}\frac{\mathrm{d}M_0^2}{\mathrm{d}\zeta}\frac{\mathrm{d}k_1}{\mathrm{d}\zeta} + \left[1 - \frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\left(\xi - \frac{M_0^2}{\sigma+2}\right)\right]k_1 = 0 \qquad \text{at } \zeta = 0 \tag{11a}$$

This condition, therefore, relates the first derivative of k_1 to k_1 at a sonic throat.

A second boundary condition results from prescribing the transverse pressure gradient $\epsilon\partial p_1/\partial x$ at an initial (upstream) location $\zeta = \zeta_i$; hence, the condition

$$k_1 = \frac{R^*}{p_0^*}\frac{\partial p_1}{\partial x} \qquad \text{at } \zeta = \zeta_i \tag{11b}$$

In applications to rocket motors, the transverse pressure gradient at the entrance to the nozzle needs to be determined from pressure matching with the flowfield in the motor chamber adjacent to the nozzle. Since the combustion gas velocity in the chamber is typically far below the speed of sound, the calculation is simplified by the assumption of incompressibility, although it is sometimes complicated by the geometry of the boundaries. This latter calculation is outside the scope of the present paper.

After $k_1(\zeta)$ has been obtained from a solution of Eq. (9), the transverse pressure gradient is obtained from Eqs. (7) and (8). By integration, one obtains the asymmetric force ϵF_1 (per unit axial length) exerted by the gas on the nozzle wall. This force is often the quantity of greatest interest in applications. For

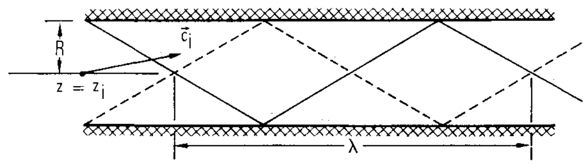

Fig. 1 Supersonic flow with small angle of incidence through rectangular duct. R = half width; c_i = incident flow velocity; λ = period.

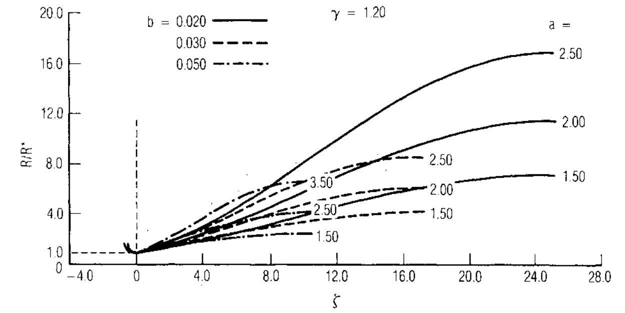

Fig. 2 Nozzle contours satisfying Eq. (14), for $\gamma = 1.20$.

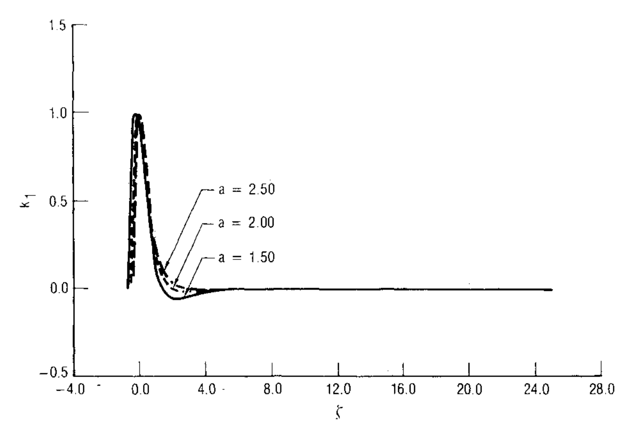

Fig. 3 Function $k_1(\zeta)$ computed from Eq. (9) and normalized to $k_1 = 1$ at nozzle throat.

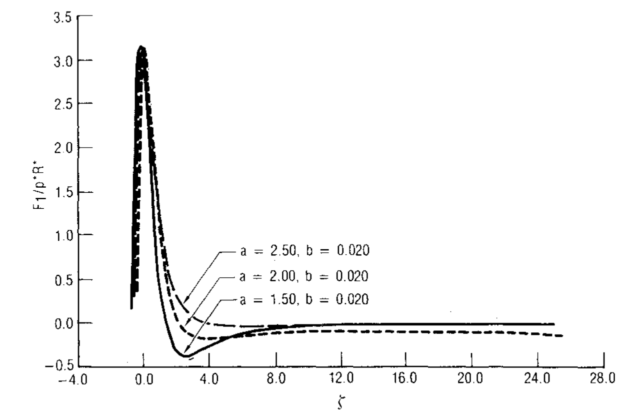

Fig. 4 Nondimensional ratio $F_1(\zeta)/(p^*R^*)$ of the transverse force F_1 on the nozzle wall, per unit axial length.

instance, for the axisymmetric nozzle, after integrating over the polar angle ϕ

$$F_1 = \pi(p_0^*/R^*)\, R^2(\zeta)\, k_1(\zeta) \tag{12}$$

Numerical results for the homogeneous case, applied to a parameterized family of nozzle contours, are discussed in Section V.

IV. Test Case

Supersonic flow, incident at a small angle on a rectangular duct of uniform width (Fig. 1), represents an example in which an exact—at least in the sense of a first-order perturbation result—solution is easily calculated. It is of interest, therefore, to compare, as a test case, results obtained from Eq. (9) with the exact, but more restricted, solution in this special case. The exact first-order solution, which is well-known, is described by a periodic pattern of triangular and rhombic regions bounded by Mach lines, in which the flow properties are constant but changing discontinuously across the Mach lines.

With M_0 again designating the unperturbed Mach number, and R the half-width, the period λ of the flow perturbation is given by

$$\lambda = 4R\sqrt{M_0^2 - 1} \qquad \text{(exact first-order theory)} \tag{13a}$$

On the other hand, from Eq. (9), since here $d\xi/d\zeta = dM_0^2/d\zeta = 0$ and $\sigma = 1$

$$\frac{d^2 k_1}{d\zeta^2} + \frac{3}{\xi^2(M_0^2 - 1)} k_1 = 0$$

The transverse pressure gradient, and similarly the other perturbation quantities, therefore, have a sinusoidal dependence on the axial coordinate, with period

$$\lambda = \frac{2\pi}{\sqrt{3}} R\sqrt{M_0^2 - 1} = 3.628\, R\sqrt{M_0^2 - 1} \qquad \text{(integral method)} \tag{13b}$$

The integral method, therefore, gives the correct Mach number dependence, although with a multiplier that differs from the correct one by approximately 10%.

V. Nozzles with Polynomial Mach Number Dependence

As an example of the application of Eq. (9), we consider an axisymmetric Laval nozzle for which the square of the unperturbed (quasi-one-dimensional) Mach number varies with axial distance as a second-degree polynomial, for which we write

$$M_0^2 - 1 = a\zeta(1 - b\zeta), \qquad a > 0,\ b \geq 0 \tag{14}$$

where a and b are constants. The nozzle contour is easily calculated from Eq. (10). For suitably chosen constants, Eq. (14) results in quite realistic nozzle contours (Fig. 2).

At $M_0 = 0$, the slope of the nozzle contour becomes infinite. This occurs at $\zeta = -(2b)^{-1}(\sqrt{1 + 4b/a} - 1)$. Clearly, the assumption of quasi-one-dimensional flow breaks down for ζ approaching this value. At $\zeta = (2b)^{-1}$, the Mach number M_0 and, hence, the nozzle radius each reach a maximum, which occurs in the supersonic part of the flow.

The transverse pressure gradient in the form of the function $k_1(\zeta)$ and the lateral force per unit axial length exerted on the nozzle, in the form of the nondimensional ratio $F_1(\zeta)/(p^*R^*)$, are computed for several values of the nozzle parameters a and b. They are graphed in Figs. 3 and 4. (For the range of b that is of practical interest for realistic nozzle contours, the curves in Fig. 3 very nearly coincide and depend only on the parameter a.) The solutions are computed for a ratio of the specific heats $\gamma = 1.20$ (a value that is representative of many rocket motor combustion gases) and are normalized so that $k_1 = 1$ at the nozzle throat. It is evident that the largest transverse pressure gradients and nozzle side forces occur in the subsonic and transonic sections of the nozzle, where most of the readjustment of the flow direction takes place. It is also evident, particularly for the smaller values of the parameter a, that the transverse pressure gradient downstream of the throat at first reverses sign, an indication of the reflection on the nozzle walls of the compression and rarefaction waves associated with the turning of the flow.

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